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# Solution of the eigenvalue problem of a $N$-level system coupled to a bosonic degree of freedom without using RWA

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**Abstract.** The eigenvalue problem of a $N$-level system coupled to a bosonic degree of freedom is solved without using RWA. For that purpose, the bosonic degree of freedom is transformed to Bargmann's Hilbert space of analytical functions. In this representation the Schrödinger equation is a system of $N$ coupled linear differential equations of first order. Using a discrete symmetry, these equations are simplified by a suitable transformation of the independent variable. Starting from the simplified equations, we develop a method to solve the eigenvalue problem of the $N$-level system. In addition, we present a simple approximate treatment and compare it with the exact results. The approximation turns out to be quite good up to outer level resonance and can be used to explain the differently structured regions in the energy spectra.

## 1. Introduction

In this paper we deal with the eigenvalue problem of the Hamiltonian

$$\hat{H} = b^+ b + \tfrac{1}{2} + \sqrt{2}\kappa \hat{J}_x (b^+ + b) + \delta \hat{J}_z. \tag{1.1}$$

Here $b^+$ and $b$ denote the creation and annihilation operators of the boson. $\hat{J}_x$, $\hat{J}_z$ are components of the total angular momentum which satisfy the commutation relation

$$[\hat{J}_x, \hat{J}_y] = \mathrm{i}\hat{J}_z. \tag{1.2}$$

The Hamiltonian (1.1) can be interpreted in two ways. For a $N$-dimensional representation of the operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$, the Hamiltonian describes the interaction of a linearly polarised mode of quantised electromagnetic field with a $N$-level atom in linear approximation. The distance between adjacent levels of the atom is always $\delta$. Therefore the atom and the field are in resonance for $\delta = 1$. In addition to 'normal' resonance we shall speak of 'outer level resonance' when $\delta$ is $1/(N-1)$. In a second interpretation, the Hamiltonian (1.1) can be used to discuss the interaction of a quantised field mode with $N-1$ two-level atoms. For this purpose we write the collective atomic operators $\hat{J}_i$ in terms of the single atomic operators $\frac{1}{2}\sigma_i$ (the $\sigma_i$ are simply the Pauli matrices):

$$\hat{J}_i = \tfrac{1}{2} \sum_{n=1}^{N-1} \sigma_i^{(n)}. \tag{1.3}$$

The collective atomic states can be expressed in terms of Dicke states (Dicke 1954) which are defined as follows:

$$\begin{aligned} \hat{J}^2 |j, m\rangle &= j(j+1)|j, m\rangle \\ \hat{J}_z |j, m\rangle &= m|j, m\rangle. \end{aligned} \tag{1.4}$$

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In this case $j$ is the cooperation number and a good quantum number since

$$[\hat{H}, \hat{J}^2] = 0. \tag{1.5}$$

If we look only at the linear space $L$ of states built up by the $N$ Dicke states

$$|(N-1)/2, m\rangle \qquad m = -(N-1)/2, \ldots, (N-1)/2 \tag{1.6}$$

we find the same $N$-dimensional representation of the operators $\hat{J}_i$ as above. By application of the Hamiltonian (1.1) the space $L$ is not left. The development of an initial state in $L$ is determined by the Hamiltonian (1.1).

The eigenvalue problem of the RWA Hamiltonian

$$\hat{H}_{\mathrm{RWA}} = b^+ b + \tfrac{1}{2} + \sqrt{2}\kappa(b\hat{J}_+ + b^+\hat{J}_-) + \delta\hat{J}_z \tag{1.7}$$

was discussed long ago (Tavis and Cummings 1968, 1969, Mallory 1969, Scharf 1970a, b). In the case of resonance ($\delta = 1$) and small coupling ($\kappa \ll 1$) the RWA Hamiltonian is a good approximation to the Hamiltonian (1.1). Recently, experiments have been carried out on Rydberg atom masers (Kaluzny *et al* 1983, Raimond *et al* 1982, Goy *et al* 1982, 1983, Moi *et al* 1983, Meschede *et al* 1985). The coupling constant $\kappa$ is proportional to the square of the principal quantum number $n$. For Rydberg atoms $n$ lies between 20 and 50 and therefore $\kappa$ is about three orders of magnitude larger than in low excited systems. For this reason we compare the energy spectrum of (1.1) with that of the RWA Hamiltonian, thereby testing the validity of RWA as a function of $\kappa$.

The Hamiltonian (1.1) is also of interest in another context. It was shown that the semiclassical equations which correspond to (1.1) lead to chaotic motion (Belobrov *et al* 1976, Miloni *et al* 1983). Therefore, as pointed out by Ackerhalt *et al* (1985), a fully quantised treatment of the $N$-level system would be interesting in connection with the discussion of 'quantum chaos'. For $N = 2$, Kus (1985) has considered the statistical properties of the energy spectrum to investigate this question. We will not discuss this point further. Instead we present the results of the fully quantised calculation. Our treatment will be similar to the treatment of the two-level atom system by Reik *et al* (1985) and Klenner *et al* (1986). The main difference is that we arrive at three-term vector recurrence relations instead of three-term scalar ones in the two-level case.

The methods we have used here to solve the eigenvalue problem of the Hamiltonian (1.1) turned out to be highly suitable for the calculation of exact isolated solutions of similar quantum optical systems (Kus and Lewenstein 1986).

This paper is organised as follows. In § 2 the transformation of the bosonic degree of freedom to Bargmann's Hilbert space of analytical functions will be described (Bargmann 1961, 1962). Then the Schrödinger equation consists of a system of $N$ coupled linear differential equations of first order. A discrete symmetry will be used to introduce a new independent variable. A solution of the eigenvalue problem of (1.1) will be given in § 3. In § 4, we shall present a simple approximate treatment of the $N$-level system. It works well for $0 < \delta < 1/(N-1)$ (up to outer level resonance). Finally, we shall discuss the differently structured regions in the energy spectra by means of the states which are used in the approximate treatment in § 5. This will lead to qualitative understanding of the $N$-level system up to resonance ($\delta = 1$).

## 2. Formulation of the $N$-level Hamiltonian in Bargmann space: separation of the positive and negative parity states

### *2.1. The N-level Hamiltonian in Bargmann space*

We introduce the Bargmann space method to treat the bosonic degree of freedom. It has already been used in the case $N=2$ by Schweber (1967), Reik *et al* (1982, 1985) and Klenner *et al* (1986) to discuss the eigenvalue problem and the dynamics of the Hamiltonian (1.1).

The transformation of the boson degree of freedom to Bargmann's Hilbert space of analytical functions is as follows. The eigenstates of the number operator are mapped onto the powers of a complex variable $\zeta$:

$$(b^+)^n|0\rangle \to \zeta^n. \tag{2.1}$$

Consequently one obtains

$$b^+ \to \zeta \qquad b \to \mathrm{d}/\mathrm{d}\zeta \tag{2.2}$$

so that the Hamiltonian and the eigenfunctions are written as

$$\hat{H} = \zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} + \frac{1}{2} + \sqrt{2}\kappa\hat{J}_x\left(\zeta + \frac{\mathrm{d}}{\mathrm{d}\zeta}\right) + \delta\hat{J}_z \tag{2.3}$$

$$|\psi\rangle = \sum_{m=-j}^{+j} \phi_m(\zeta)|j, m\rangle. \tag{2.4}$$

Here we choose the states $|j, m\rangle$ to be the eigenstates of the operator $\hat{J}_x$ ($\hat{J}_x$ is diagonal):

$$\hat{J}_x|j, m\rangle = m|j, m\rangle \qquad m = -j, -j+1, \ldots, j \tag{2.5}$$

and $j$ is $(N-1)/2$.

Since $\hat{J}_x$ and $\hat{J}_z$ are components of the total angular momentum operator, the action of $\hat{J}_z$ is

$$\hat{J}_z|j, m\rangle = \tfrac{1}{2}g_m^{(-)}|j, m-1\rangle + \tfrac{1}{2}g_m^{(+)}|j, m+1\rangle \tag{2.6}$$

with

$$g_m^{(-)} = [(j+m)(j-m+1)]^{1/2}$$
$$g_m^{(+)} = [(j-m)(j+m+1)]^{1/2}.$$

The Schrödinger equation

$$\hat{H}|\psi\rangle = \lambda|\psi\rangle \tag{2.7}$$

after collecting the components of $|j, m\rangle$ ($m = -j, -j+1, \ldots, j$), consists of $N = 2j+1$ coupled differential equations of first order:

$$\left(\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} + \tfrac{1}{2} - \lambda\right)\phi_m(\zeta) + \sqrt{2}\kappa\left(\zeta + \frac{\mathrm{d}}{\mathrm{d}\zeta}\right)m\phi_m(\zeta) + \tfrac{1}{2}\delta(g_{m-1}^{(+)}\phi_{m-1}(\zeta) + g_{m+1}^{(-)}\phi_{m+1}(\zeta)) = 0. \tag{2.8}$$

The component functions $\phi_m(\zeta)$ have to belong to the space of entire functions. These have a finite norm with respect to the inner product in Bargmann space defined by

$$(f|g) = \frac{1}{\pi}\int_{\mathbb{C}} \exp(-|\zeta|^2)f(\zeta^+)g(\zeta)\,\mathrm{d}\,\mathrm{Re}\,\zeta\,\mathrm{d}\,\mathrm{Im}\,\zeta. \tag{2.9}$$

This requirement determines the eigenvalues. The main advantage of the Bargmann space treatment lies in the fact that the Schrödinger equation (2.8) is a system of linear differential equations of first order only. This is due to the linear coupling of the atoms to the quantised field.

### *2.2. Discrete symmetry of the N-level Hamiltonian: transformation of the independent variable*

The Hamiltonian (1.1) has a constant of motion which is given by

$$\hat{P} = \exp\left[ i\pi\left( \zeta \frac{d}{d\zeta} + \tfrac{1}{2}(\hat{A} - 1) \right) \right] \tag{2.10}$$

and corresponds to a discrete symmetry. The operator $A$ does not act on the bosonic degree of freedom and is given by

$$\hat{A} = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & & \iddots & 0 \\ 0 & 1 & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \tag{2.11}$$

in the basis $|j, m\rangle$. Since $\hat{P}^2 = 1$, the spectrum of $\hat{P}$ is $\pm 1$. Therefore, the eigenfunctions of the Hamiltonian can be labelled by the eigenvalues $\pm 1$ of the parity operator $\hat{P}$.

The existence of the parity $\hat{P}$ as a constant of motion leads to a condition for the component functions $\phi_m(\zeta)$. From

$$\begin{aligned} \hat{P}|\psi\rangle &= \hat{P}\left( \sum_{m=-j}^{+j} \phi_m(\zeta)|j, m\rangle \right) \\ &= \sum_{m=-j}^{+j} \phi_{-m}(-\zeta)|j, m\rangle \\ &= \pm \sum_{m=-j}^{+j} \phi_m(\zeta)|j, m\rangle = \pm|\psi\rangle \end{aligned} \tag{2.12}$$

we derive

$$\phi_m(\zeta) = \pm\phi_{-m}(-\zeta). \tag{2.13}$$

The upper (lower) sign stands for positive (negative) parity states.

Now we introduce the functions

$$\begin{aligned} F_m(\zeta) &= \phi_m(\zeta) + \phi_{-m}(\zeta) \\ G_m(\zeta) &= \phi_m(\zeta) - \phi_{-m}(\zeta). \end{aligned} \tag{2.14}$$

Equations (2.14) show that

$$F_m(\zeta) = F_{-m}(\zeta) \qquad \text{and} \qquad G_m(\zeta) = -G_{-m}(\zeta).$$

Therefore we take into account only positive values of $m$. Then $m = 0, 1, 2, \dots, j$ for $N$ odd and $m = \frac{1}{2}, \frac{3}{2}, \dots, j$ for $N$ even. Note that

$$\begin{aligned} F_m(\zeta) &= \pm F_m(-\zeta) \\ G_m(\zeta) &= \mp G_m(-\zeta). \end{aligned} \tag{2.15}$$

This means that $F_m(\zeta)$ is an even (odd) function of $\zeta$ and $G_m(\zeta)$ is an odd (even) function of $\zeta$ for positive (negative) parity. Using the differential equations (2.8) we find the equations for the functions just introduced $F_m(\zeta)$ and $G_m(\zeta)$ to be

$$\left(\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}+\tfrac{1}{2}-\lambda\right)F_m(\zeta)+\sqrt{2}\kappa m\left(\zeta+\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)G_m(\zeta)+\tfrac{1}{2}\delta(g^{(+)}_{m-1}F_{m-1}(\zeta)+g^{(-)}_{m+1}F_{m+1}(\zeta))=0$$

$$\left(\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}+\tfrac{1}{2}-\lambda\right)G_m(\zeta)+\sqrt{2}\kappa m\left(\zeta+\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)F_m(\zeta)+\tfrac{1}{2}\delta(g^{(+)}_{m-1}G_{m-1}(\zeta)+g^{(-)}_{m+1}G_{m+1}(\zeta))=0. \tag{2.16}$$

Now we restrict ourselves to the states of positive parity. In this case, the functions $F_m(\zeta)$ and $H_m(\zeta)=G_m(\zeta)/\zeta$ are even functions of $\zeta$. For this reason, we introduce the new variable

$$z=\zeta^2 \tag{2.17}$$

and transform the differential equations (2.16) to

$$\left(2z\frac{\mathrm{d}}{\mathrm{d}z}+\tfrac{1}{2}-\lambda\right)F_m(z)+\sqrt{2}\kappa m\left(z+2z\frac{\mathrm{d}}{\mathrm{d}z}+1\right)H_m(z)$$
$$+\tfrac{1}{2}\delta(g^{(+)}_{m-1}F_{m-1}(z)+g^{(-)}_{m+1}F_{m+1}(z))=0$$

$$\left(2z\frac{\mathrm{d}}{\mathrm{d}z}+\tfrac{3}{2}-\lambda\right)H_m(z)+\sqrt{2}\kappa m\left(1+2\frac{\mathrm{d}}{\mathrm{d}z}\right)F_m(z)+\tfrac{1}{2}\delta(g^{(+)}_{m-1}H_{m-1}(z)+g^{(-)}_{m+1}H_{m+1}(z))=0. \tag{2.18}$$

Compared with the differential equations (2.16) or (2.8), the set (2.18) has the advantage of leading to much simpler recurrence relations. This point will be discussed in more detail in § 3. Additionally we will see how the positive and negative parity solutions can be obtained from equations (2.18).

## 3. Solution of the eigenvalue problem of the $N$-level system

### *3.1. N-dimensional two-term vector recurrence relations*

The advantage of using the differential equations (2.18) to solve the eigenvalue problem of the $N$-level system is that they lead to a $N$-dimensional two-term vector recurrence relation as we show in this section. This is equivalent to a three-term $(N/2)$-dimensional vector recurrence relation. In contrast to this, an expansion of the functions $\phi_m(\zeta)$ in the variable $\zeta$ to solve the differential equations (2.8) leads to a $N$-dimensional three-term recurrence relation. In a sense, the dimensionality of the problem is reduced by a factor of two when equations (2.18) are used. The insertion of the expansion

$$F_m(z)=\sum_{n=0}^{\infty}f_n^m z^{n+s}$$
$$H_m(z)=\sum_{n=0}^{\infty}h_n^m z^{n+s} \tag{3.1}$$

in equations (2.18) leads to the following recurrence relations:

$$[2(n+s)+\tfrac{1}{2}-\lambda]f_n^m+\sqrt{2}\kappa m[2(n+s)+1]h_n^m+\sqrt{2}\kappa m h_{n-1}^m$$
$$+\tfrac{1}{2}\delta g^{(+)}_{m-1}f_n^{m-1}+\tfrac{1}{2}\delta g^{(-)}_{m+1}f_n^{m+1}=0$$
$$[2(n+s)+\tfrac{3}{2}-\lambda]h_n^m+\sqrt{2}\kappa m f_n^m+2\sqrt{2}\kappa m(n+s+1)f_{n+1}^m+\tfrac{1}{2}\delta g^{(+)}_{m-1}h_n^{m-1}$$
$$+\tfrac{1}{2}\delta g^{(-)}_{m+1}h_n^{m+1}=0. \tag{3.2}$$

As we mentioned before $f_n^m = f_n^{-m}$ and $h_n^m = -h_n^{-m}$. Now assume $N$ to be even. In this case, equations (3.2) can be written as a $N$-dimensional two-term vector recurrence relation:

$$\begin{pmatrix} \hat{M}_{11}(n+1) & \hat{M}_{12}(n+1) \\ \hat{M}_{21}(n+1) & \hat{M}_{22}(n+1) \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_{n+1} \\ \boldsymbol{h}_{n+1} \end{pmatrix} + \begin{pmatrix} \hat{N}_{11}(n) & \hat{N}_{12}(n) \\ \hat{N}_{21}(n) & \hat{N}_{22}(n) \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_n \\ \boldsymbol{h}_n \end{pmatrix} = 0. \tag{3.3}$$

The $\hat{M}_{ik}(n+1)$ and $\hat{N}_{ik}(n)$ are $(N/2)$-dimensional matrices and

$$\boldsymbol{f}_n = \begin{pmatrix} f_n^{1/2} \\ f_n^{3/2} \\ f_n^{5/2} \\ \vdots \\ f_n^{j} \end{pmatrix} \qquad \boldsymbol{h}_n = \begin{pmatrix} h_n^{1/2} \\ h_n^{3/2} \\ h_n^{5/2} \\ \vdots \\ h_n^{j} \end{pmatrix}. \tag{3.4}$$

The matrices $\hat{M}_{ik}(n+1)$ and $\hat{N}_{ik}(n)$ have a simple structure:

$$[\hat{M}_{11}(n+1)]_{ij} = [2(n+s+1)+\tfrac{1}{2}-\lambda+\delta g_{1/2}^{(-)}\Delta_i/2]\,\delta_{ij} + \tfrac{1}{2}\,\delta g_{j-1/2}^{(-)}\,\delta_{ij-1} + \tfrac{1}{2}\,\delta g_{j-1/2}^{(+)}\,\delta_{ij+1}.$$

Here

$$\Delta_i = \begin{cases} 1 & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover we find

$$\begin{aligned} (\hat{M}_{12}(n+1))_{ij} &= \sqrt{2}\kappa[2(n+s+1)+1](i-\tfrac{1}{2})\,\delta_{ij} \\ (\hat{M}_{21}(n+1))_{ij} &= \sqrt{2}\kappa[2(n+s+1)](i-\tfrac{1}{2})\,\delta_{ij} \\ (\hat{M}_{22}(n+1))_{ij} &= 0 \\ (\hat{N}_{11}(n))_{ij} &= 0 \\ (\hat{N}_{12}(n))_{ij} &= \sqrt{2}\kappa(i-\tfrac{1}{2})\,\delta_{ij} \\ (\hat{N}_{21}(n))_{ij} &= \sqrt{2}\kappa(i-\tfrac{1}{2})\,\delta_{ij} \\ (\hat{N}_{22}(n))_{ij} &= (\hat{M}_{11}(n, \lambda-1))_{ij}. \end{aligned} \tag{3.5}$$

Except for $\hat{M}_{11}(n+1)$ and $\hat{N}_{22}(n)$, all these matrices are diagonal. The indicial equation, which determines the lowest order coefficients of the expansion (3.1), becomes

$$\begin{pmatrix} \hat{M}_{11}(0) & \hat{M}_{12}(0) \\ \hat{M}_{21}(0) & \hat{M}_{22}(0) \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_0 \\ \boldsymbol{h}_0 \end{pmatrix} = 0. \tag{3.6}$$

A non-trivial solution of (3.6) exists, when the determinant of the matrix of the LHS is zero. This is the case for $s = 0$ and $s = -\frac{1}{2}$ as can be seen from equations (3.5). The solution is not unique for both values of $s$. There are $N/2$ linear independent solutions of equation (3.6) in both cases. What changes now when $N$ is odd? Then the first equation (3.2) combines only the expansion coefficients of $z^{n+s}$ and the second equation vanishes identically for $m = 0$. Therefore, equations (3.2) result in a $(N-1)$-dimensional two-term recurrence relation. Since $(N-1)$ is even, this vector recurrence relation takes the same form as equation (3.3). So, there is no change in the analysis when $N$ is odd.

### *3.2. Solutions of both parities*

Now we discuss the physical relevance of the solutions corresponding to the two values of $s$. As mentioned, the functions $F_m(\zeta)$ and $G_m(\zeta) = \zeta H_m(\zeta)$ should be entire functions in $\zeta$. For $s = 0$, there is no branch point and no pole of $F_m(\zeta)$ and $G_m(\zeta)$ at the origin. For $s = -\frac{1}{2}$ it seems that the expansion of $F_m(\zeta)$ starts with $1/\zeta$ in contrast to the expansion of $G_m(\zeta)$ which starts with $\zeta^0$, but this is prevented by the initial condition. For $s = -\frac{1}{2}$ we have $\hat{M}_{12}(0) = 0$ and the initial condition is

$$f_0 = 0 \qquad \boldsymbol{h}_0 \text{ arbitrary.} \tag{3.7}$$

Therefore, the $F_m(\zeta)$ starts with $\zeta^0$ and similar to $s = 0$ there is no branch point and no pole at the origin. Provided that the expansion (3.1) is convergent in the whole $\zeta$ plane, both values of $s$ lead to a physical solution.

We now show that positive parity states result from $s = 0$, whereas the negative ones arise from $s = -\frac{1}{2}$. This becomes clear when the expansion (3.1) is written in the variable $\zeta$, also taking into account equation (2.15):

$$\begin{aligned} s &= 0 & s &= -\tfrac{1}{2} \\ F_m(\zeta) &= \sum_{n=0}^{\infty} f_n^m \zeta^{2n} & F_m(\zeta) &= \sum_{n=1}^{\infty} f_n^m \zeta^{2n-1} \\ H_m(\zeta) &= \sum_{n=0}^{\infty} h_n^m \zeta^{2n+1} & H_m(\zeta) &= \sum_{n=0}^{\infty} h_n^m \zeta^{2n}. \end{aligned} \tag{3.8}$$

Although initially constructed for positive parity states only, equation (2.18) now leads to solutions of both parity.

### *3.3. Solution of the eigenvalue problem*

We are going to describe the procedure used to solve equations (3.3). In order to motivate it, we eliminate $\boldsymbol{h}_{n+1}$ and $\boldsymbol{h}_n$ from (3.3) to obtain the $(N/2)$-dimensional vector recurrence relation

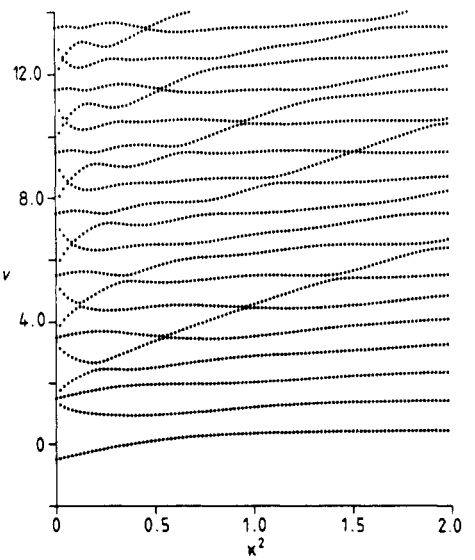


**Figure 1.** Energy spectrum of $N$-level system for positive parity as a function of the square of the coupling constant for $N = 3$ and $\delta = 1$.

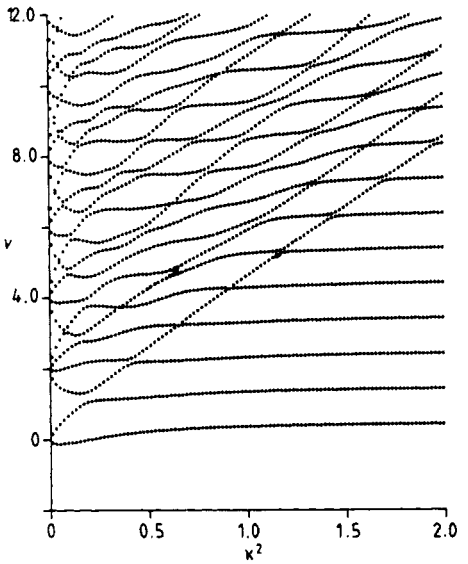


**Figure 2.** As figure 1, but for $N = 4$.

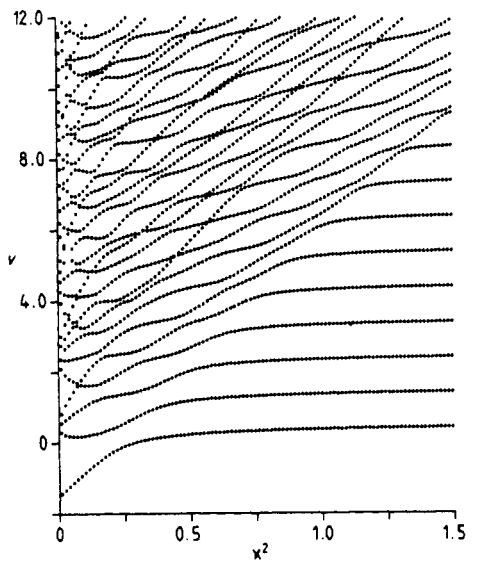


**Figure 3.** As figure 1, but for $N = 5$.

$$\hat{A}_1(n)f_{n+2} + \hat{A}_2(n)f_{n+1} + \hat{A}_3(n)f_n = 0 \tag{3.9}$$

with

$$\hat{A}_1(n) = \hat{M}_{12}(n+1)\hat{N}_{22}^{-1}(n+1)\hat{M}_{21}(n+2)$$

$$\hat{A}_2(n) = \hat{M}_{12}(n+1)\hat{N}_{22}^{-1}(n+1)\hat{N}_{21}(n+1) + \hat{N}_{12}(n)\hat{N}_{22}^{-1}(n)\hat{M}_{21}(n+1) - \hat{M}_{11}(n+1)$$

and

$$\hat{A}_3(n) = \hat{N}_{12}(n)\hat{N}_{22}^{-1}(n)\hat{N}_{21}(n).$$

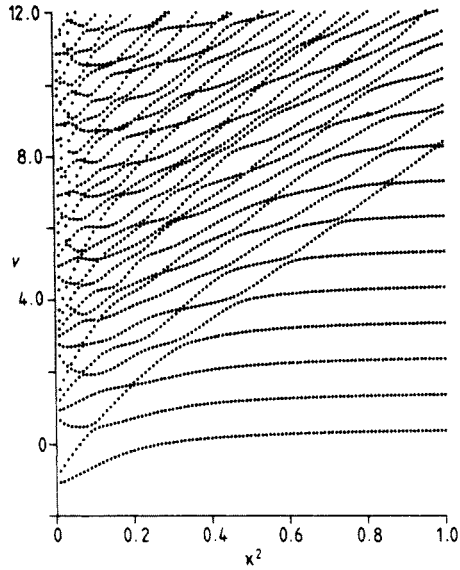


**Figure 4.** As figure 1, but for $N = 6$.

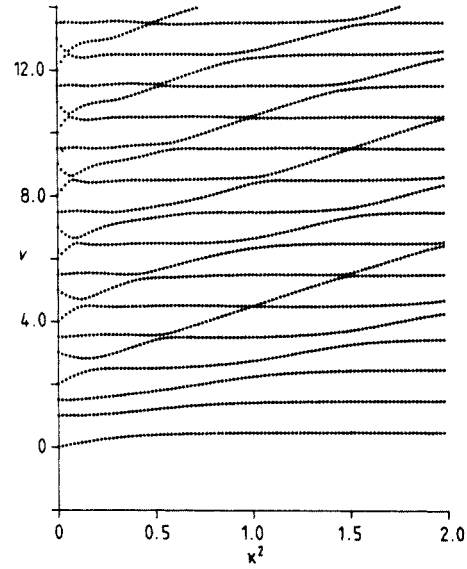


**Figure 5.** As figure 1, but for $\delta = \frac{1}{2}$.

This recurrence relation is assumed to terminate for a given value of $n$:

$$f_{n+i} = 0 \qquad i = 0, 1, 2, \ldots.$$

This leads to

$$\begin{aligned} &\hat{A}_3(n-1)f_{n-1} = 0 \\ &\hat{A}_2(n-2)f_{n-1} + \hat{A}_3(n-2)f_{n-2} = 0 \\ &\quad\vdots \qquad\qquad\qquad\quad \vdots \end{aligned} \tag{3.10}$$

If $\hat{A}_3(n-1) \neq 0$, all $f_j$ have to be zero—but this is the trivial solution only. On the other hand, if $\hat{A}_3(n-1) = 0$, this leads to an exact non-trivial solution. The recurrence relation

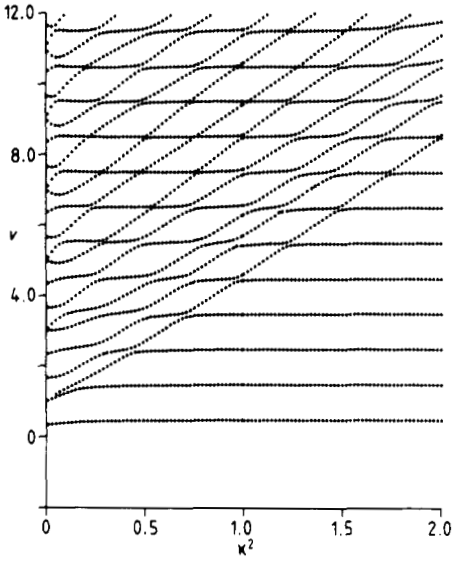


**Figure 6.** As figure 2, but for $\delta = \frac{1}{3}$.

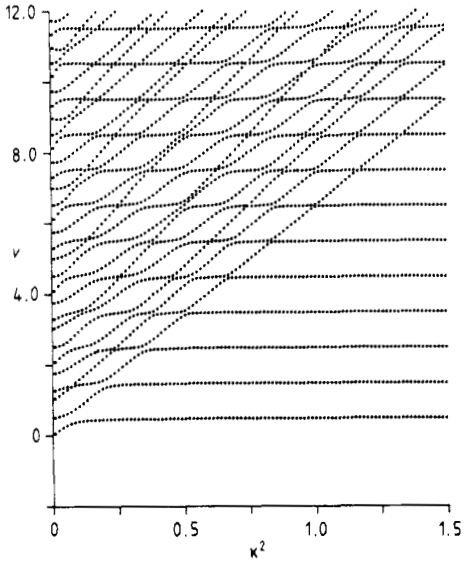


**Figure 7.** As figure 3, but for $\delta = \frac{1}{4}$.

has the following infinite behaviour:

$$\lim_{n\to\infty} \hat{A}_3(n) = 0.$$

So, neglecting the first equation of (3.10) for sufficiently large $n$, an approximate solution of the eigenvalue problem of the $N$-level system is obtained. The second equation of (3.10), equivalent to

$$f_n = 0 \tag{3.11}$$

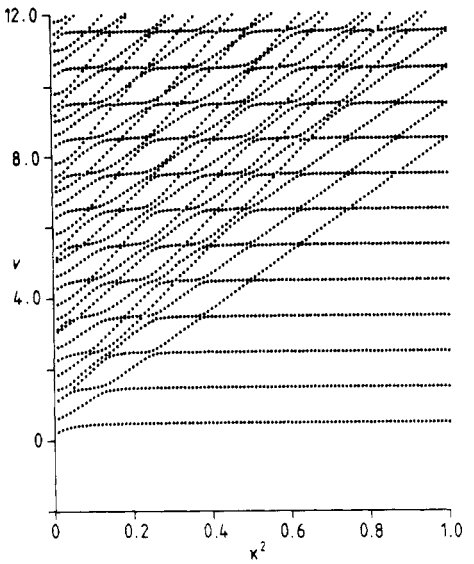


**Figure 8.** As figure 4, but for $\delta = \frac{1}{5}$.

is a condition to terminate the recurrence relation. Our procedure is the following: the recurrence relations (3.3) are written as

$$\begin{pmatrix} \boldsymbol{f}_{n+1} \\ \boldsymbol{h}_{n+1} \end{pmatrix} = \begin{pmatrix} \hat{R}_{11}(n) & \hat{R}_{12}(n) \\ \hat{R}_{21}(n) & \hat{R}_{22}(n) \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_n \\ \boldsymbol{h}_n \end{pmatrix}. \tag{3.12}$$

The matrices $\hat{R}_{ij}(n)$ can be calculated explicitly and are given in the appendix. As mentioned before, there are $N/2$ linear independent solutions of the indicial equation (3.6):

$$\begin{pmatrix} \boldsymbol{f}_0 \\ \boldsymbol{h}_0 \end{pmatrix}_i \qquad i = 1, 2, \ldots, N/2. \tag{3.13}$$

The vectors

$$\begin{pmatrix} \boldsymbol{f}_{n+1} \\ \boldsymbol{h}_{n+1} \end{pmatrix}_i \qquad i = 1, 2, \ldots, N/2 \tag{3.14}$$

are calculated by repeated application of (3.12). They are still linearly independent, because the determinant of the matrix in equation (3.12) is unequal zero for all $n$. As $\boldsymbol{f}_{n+1}$ should be zero, the vectors (3.14) have to form a linear combination satisfying

$$\sum_{i=1}^{N/2} a_i (\boldsymbol{f}_{n+1})_i = 0. \tag{3.15}$$

In other words, the initial vectors (3.13) have to be combined in such a way that the repeated application of (3.12) leads to

$$\boldsymbol{f}_{n+1} = 0. \tag{3.16}$$

The condition for a non-trivial solution of (3.15)—that means $a_1, a_2, \ldots, a_{N/2} \neq 0$—is

$$\det((\boldsymbol{f}_{n+1})_1 (\boldsymbol{f}_{n+1})_2 \ldots (\boldsymbol{f}_{n+1})_{N/2}) = 0. \tag{3.17}$$

Since this is a function of $\kappa$, $\delta$ and the energy $\lambda$, it determines the energy eigenvalues.

**Table 1.** Accurate decimal places of the energy eigenvalues. Used parameters: $N=4$, $\delta=1$, $\kappa=1$.

| | $n$ | | | |
|---|---|---|---|---|
| $i$ | 15 | 20 | 25 | 30 |
| 0 | 12 | 12 | 12 | 12 |
| 5 | 7 | 12 | 12 | 12 |
| 10 | 4 | 9 | 12 | 12 |
| 15 | 1 | 6 | 12 | 12 |
| 20 | 0 | 4 | 11 | 12 |
| 25 | 0 | 2 | 8 | 12 |

The accuracy of this procedure depends on the number of iterations $n$, the excited state $i$ and the parameters used, $\kappa$ and $\delta$. Table 1 displays this dependence for $N=4$ and a typical combination of the parameters $\kappa$ and $\delta$.

We calculated the energy spectrum of positive parity for $N=3, 4, 5$ and 6 as a function of $\kappa^2$. Figures 1-4 present the energy spectra for 'normal' resonance ($\delta=1$) whereas figures 5-8 show those for 'outer level resonance' ($\delta=1/(N-1)$). For reasons of better representation we plotted $v=\lambda+2\kappa^2 j^2$ against $\kappa^2$.

## 4. A simple approximative treatment of the eigenvalue problem of the $N$-level system

In this section a simple approximate treatment of the $N$-level system is developed on the basis of the exact solution for $\delta=0$. It is a generalisation of a perturbative treatment which has already been worked out for $N=2$ by Graham and Höhnerbach (1984).

The Hamiltonian (1.1) can be written in Bargmann space as (see § 2)

$$\hat{H}=\hat{H}_0+\delta\hat{J}_z$$

with

$$\hat{H}_0=\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}+\frac{1}{2}+\sqrt{2}\kappa\hat{J}_x\left(\zeta+\frac{\mathrm{d}}{\mathrm{d}\zeta}\right). \tag{4.1}$$

The solution of the eigenvalue problem

$$\hat{H}_0|\phi\rangle=\varepsilon|\phi\rangle \tag{4.2}$$

is given by

$$|\phi\rangle_{nm}=\phi_{nm}(\zeta)|j, m\rangle \tag{4.3}$$

$$\phi_{nm}(\zeta)=(1/\sqrt{n!})\exp(-\kappa^2m^2)(\zeta+\sqrt{2}\kappa m)^n\exp(-\sqrt{2}\kappa m\zeta) \tag{4.4}$$

$$\varepsilon_{nm}=n-2(\kappa m)^2. \tag{4.5}$$

The bosonic component function $\phi_{nm}(\zeta)$ describes the $n$th eigenstate of a displaced harmonic oscillator. The displacement in configuration space is $-2\kappa m$, the energy spectrum for a given value of $\kappa^2$ being at least two-fold degenerate.

For reasons of simplicity we restrict our calculations to the four-level atom. However, the generalisation to arbitrary level number $N$ will be discussed in § 5. For $N=4$ and $\delta=0$, $\kappa=1$ the solution (4.3)-(4.5) is illustrated in figure 9.

Each atomic wavefunction $|j, m\rangle$ determines a harmonic potential for the bosonic degree of freedom. The potentials belonging to $\pm m$ lead to the twofold degeneracy mentioned above. These degenerate eigenstates can be superposed so that they become eigenstates of the operator $\hat{P}$. The result is

$$|\chi\rangle_{nm}^{(\pm)} = (1/\sqrt{2n!}) \exp(-\kappa^2 m^2)[(\zeta+\sqrt{2}\kappa m)^n \exp(-\sqrt{2}\kappa m\zeta)|j, m\rangle$$
$$\pm(-1)^n(\zeta-\sqrt{2}\kappa m)^n \exp(\sqrt{2}\kappa m\zeta)|j, -m\rangle] \tag{4.6}$$

with $m=\frac{1}{2}, \frac{3}{2}$ and $n=0, 1, 2, \ldots$, for $N=4$.

From the construction, the states (4.6) satisfy the equations

$$\hat{H}_0|\chi\rangle_{nm}^{(\pm)} = \varepsilon_{nm}|\chi\rangle_{nm}^{(\pm)}$$
$$\hat{P}|\chi\rangle_{nm}^{(\pm)} = \pm|\chi\rangle_{nm}^{(\pm)}.$$

For an approximate calculation of the energy eigenvalues of (1.1) we choose the following ansatz:

$$|\psi\rangle_n^{(\pm)} = \alpha_n^{(\pm)}|\chi\rangle_{n1/2}^{(\pm)} + \beta_l^{(\pm)}|\chi\rangle_{l3/2}^{(\pm)}. \tag{4.7}$$

The integer $n$ labels the excitation in the $m=\pm\frac{1}{2}$ potential, whereas $l$ is a pointer for the excited state in the $m=\pm\frac{3}{2}$ potential which is energetically nearest to the $n$th excited

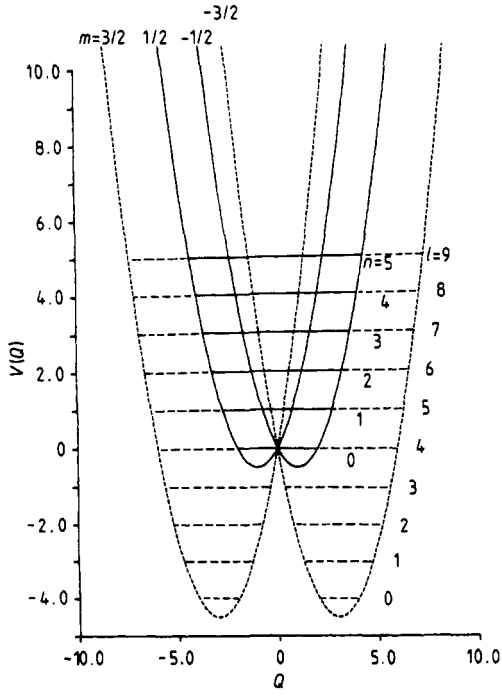


**Figure 9.** Illustration of the solution of the eigenvalue problem of the four-level system for $\delta=0$ and $\kappa=1$.

state of the inner potential. For example, for $\kappa = 1$ and $n = 1$ $l$ is equal to 5 (see figure 9). The general connection between $l$ and $\kappa^2$ (for the four-level system) is

$$l = n + k$$

$$k = \begin{pmatrix} 0 & 0 \leqslant \kappa^2 < \frac{1}{8} \\ 1 & \frac{1}{8} \leqslant \kappa^2 < \frac{3}{8} \\ 2 & \frac{3}{8} \leqslant \kappa^2 < \frac{5}{8} \\ \vdots & \vdots \quad \vdots \end{pmatrix}. \tag{4.8}$$

The insertion of the ansatz (4.7) into the Schrödinger equation

$$\hat{H}|\psi\rangle_n^{(\pm)} = \lambda|\psi\rangle_n^{(\pm)} \tag{4.9}$$

leads to

$$\begin{pmatrix} \varepsilon_{n1/2} + \delta_{n1/2}^{(\pm)}\langle\chi|\hat{J}_z|\chi\rangle_{n1/2}^{(\pm)} - \lambda & \delta_{n1/2}^{(\pm)}\langle\chi|\hat{J}_z|\chi\rangle_{l3/2}^{(\pm)} \\ \delta_{l3/2}^{(\pm)}\langle\chi|\hat{J}_z|\chi\rangle_{n1/2}^{(\pm)} & \varepsilon_{l3/2} - \lambda \end{pmatrix} \begin{pmatrix} \alpha_n^{(\pm)} \\ \beta_l^{(\pm)} \end{pmatrix} = 0. \tag{4.10}$$

In order to obtain a non-trivial solution the determinant of the matrix in (4.10) has to be zero. This condition determines the approximate energy eigenvalues:

$$\lambda_n^{(\pm)} = \pm\left[ [\delta_{n1/2}^{(\pm)}\langle\chi|\hat{J}_z|\chi\rangle_{l3/2}^{(\pm)}]^2 + \left(\frac{n - l + 4\kappa^2 + \delta_{n1/2}^{(\pm)}\langle\chi|\hat{J}_z|\chi\rangle_{n1/2}^{(\pm)}}{2}\right)^2\right]^{1/2}$$
$$+ \frac{n + l - 5\kappa^2 + \delta_{n1/2}^{(\pm)}\langle\chi|\hat{J}_z|\chi\rangle_{n1/2}^{(\pm)}}{2}. \tag{4.11}$$

A long but elementary calculation shows that

$$\delta_{n1/2}^{(\pm)}\langle\chi|\hat{J}_z|\chi\rangle_{n1/2}^{(\pm)} = \pm\tfrac{1}{2}(-1)^n(U_{nn}^{-1/2\,1/2}(\kappa) + U_{nn}^{1/2-1/2}(\kappa))$$
$$\delta_{n1/2}^{(\pm)}\langle\chi|\hat{J}_z|\chi\rangle_{l3/2}^{(\pm)} = \tfrac{1}{4}\sqrt{3}(U_{nl}^{1/2\,3/2}(\kappa) + U_{nl}^{-1/2-3/2}(\kappa)(-1)^{n+l})$$

with

$$U_{nl}^{ij}(\kappa) = (-1)^{l+n}(n!/l!)^{1/2}\exp[-\kappa^2(i-j)^2][\sqrt{2}\kappa(i-j)]^{n-l}L_n^{l-n}\{2[\kappa(i-j)]^2\} \tag{4.12}$$

where $L_n^{l-n}(x)$ is a generalised Laguerre polynomial.

**Table 2.** Comparison of exact and approximate treatments.

| | $\delta = \frac{1}{3}$ | | $\delta = 1$ | |
|---|---|---|---|---|
| Level number | $\Delta_{max}$ | $\Delta_{av}$ | $\Delta_{max}$ | $\Delta_{av}$ |
| 0 | 0.070 | 0.030 | 0.495 | 0.172 |
| 1 | 0.054 | 0.028 | 0.340 | 0.148 |
| 2 | 0.050 | 0.024 | 0.253 | 0.141 |
| 3 | 0.079 | 0.031 | 0.452 | 0.180 |
| 4 | 0.057 | 0.020 | 0.539 | 0.165 |
| 5 | 0.059 | 0.019 | 0.461 | 0.158 |
| 6 | 0.039 | 0.019 | 0.405 | 0.165 |
| 7 | 0.019 | 0.010 | 0.284 | 0.102 |
| 8 | 0.027 | 0.010 | 0.246 | 0.103 |
| 9 | 0.027 | 0.014 | 0.278 | 0.119 |
| 10 | 0.023 | 0.010 | 0.299 | 0.115 |

The approximate energy spectrum (4.11) is complete for $k = 0$ only. In general, there are $k$ states in the $m = \pm\frac{3}{2}$ potential which are not taken into account by (4.11) (see figure 9). But since $\delta\hat{J}_z$ connects only adjacent potentials, the low-lying states in the $m = \pm\frac{3}{2}$ potential are not disturbed very much. So we assume them to be given by (4.6) with the corresponding energy being (4.5).

The difference between the approximate and exact energy spectra for $N = 4$, $\delta = \frac{1}{3}$ and $\delta = 1$ is presented in table 2. Here $\Delta_{max}$ is the modulus of the maximal difference in the $\kappa^2$ range $0 \leqslant \kappa^2 \leqslant 2$. On the other hand, $\Delta_{av}$ is the average of the moduli of the differences which were calculated in the $\kappa^2$ range $0 \leqslant \kappa^2 \leqslant 1$ for $\delta = \frac{1}{3}$ and $0 \leqslant \kappa^2 \leqslant 2$ for $\delta = 1$ with a stepwidth of 0.1.

## 5. Discussion

In this paper we have presented energy spectra for the $N$-level system ($N = 3, 4, 5, 6$). The comparison of the corresponding RWA energy spectra with them shows that RWA is even in resonance ($\delta = 1$) a good approximation only for small values of $\kappa^2$ (see figures 10–13). It is interesting that the $\kappa^2$ range within which RWA is valid decreases with increasing $N$.

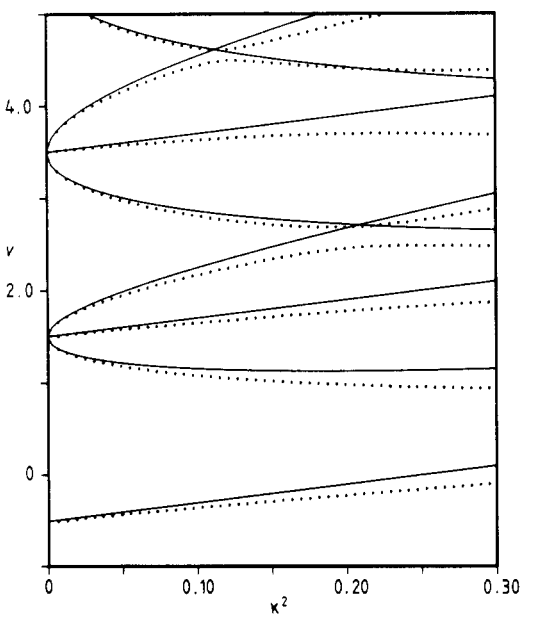


**Figure 10.** Comparison of exact and RWA energy spectra for $N = 3$, $\delta = 1$ and positive parity. (The full curves are the RWA eigenvalues.)

In § 4 we worked out a simple approximate treatment of the eigenvalue problem for the special case $N = 4$. This approximation will now be generalised for arbitrary even $N$. There are $N$ harmonic potentials which are determined by the atomic wavefunctions $|(N-1)/2, m\rangle$ for $\delta = 0$. The eigenvalues of the potentials belonging to $\pm m$ are degenerated. From the corresponding energy eigenstates, those of the parity operator $\hat{P}$ can be constructed, which are given by (4.6), but now with $m = \frac{1}{2}, \frac{3}{2}, \ldots, (N-1)/2$. Just as for $N = 4$, we assume that the first $k_1$ low-lying states of the

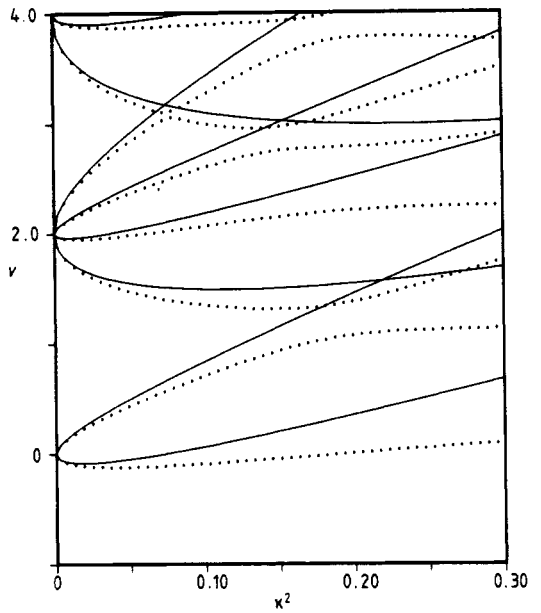


**Figure 11.** As figure 10, but for $N = 4$.

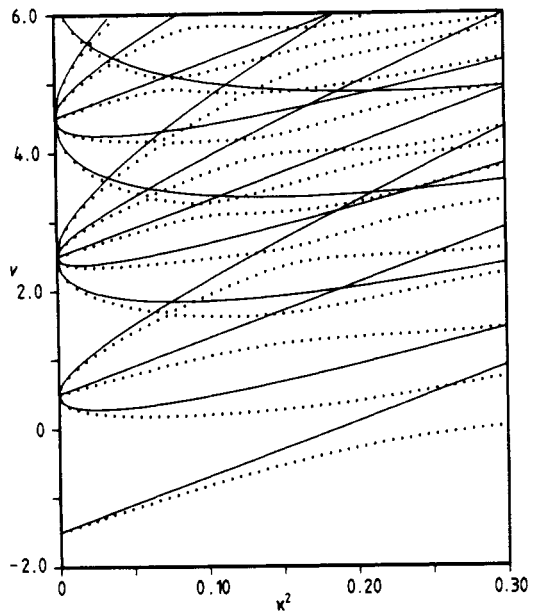


**Figure 12.** As figure 11, but for $N = 5$.

$\pm(N-1)/2$ potential are not influenced. Above, there is a region where $k_2$ states of the $\pm(N-1)/2$ potential are mixed with $k_2$ energetically neighbouring states of the $\pm[(N-1)/2-1]$ potential. In the next higher region there is an 'interaction' of three different types of states. Finally, all $N/2$ different types of states have to be taken into account, and only the energetically most neighboured, coming from different

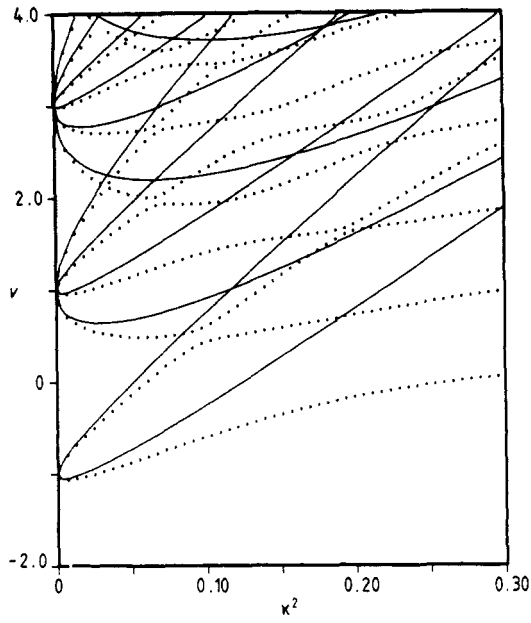


**Figure 13.** As figure 12, but for $N = 6$.

potentials, are mixing. Of course, the degree of excitation and the integers $k_1, k_2, \ldots, k_{N/2}$ are dependent on the interaction constant $\kappa$.

Summarised, the approximation leads to the following scenario: for given value of $\kappa^2$, there are $N/2$ different regions in the energy spectrum. The most low-lying region is a harmonic oscillator spectrum. In the next region two harmonic oscillator states are mixing, thus leading to a two-state structure in the energy spectrum. The region above shows three states, and so on. The size of the different regions is linearly increasing with $\kappa^2$.

We showed in § 4 that with the approximation good results for arbitrary values of $\kappa^2$ can be gained up to the outer level resonance. It is obvious from the construction that this approximation will improve for increasing values of $\kappa^2$. The different regions discussed above can easily be distinguished fom each other up to resonance (see figures 2 and 4). In this way the approximation leads to a better qualitative understanding of the eigenvalue problem of the $N$-level system. For odd $N$, an approximation can be constructed in a similar way, with the only difference being that the $m = 0$ potential is not degenerate.

## Appendix. The matrices $\hat{R}_{ij}(n)$

The matrix appearing on the LHS of equation (3.3) can be inverted explicitly because of the simple structure of the $(N/2)$-dimensional matrices $\hat{M}_{ij}(n+1)$. Therefore it is possible to calculate the $(N/2)$-dimensional matrices $\hat{R}_{ij}(n)$ of equation (3.12):

$$(\hat{R}_{11}(n))_{ij} = \tfrac{1}{2}[1/(n+s+1)]\,\delta_{ij}$$

$$(\hat{R}_{12}(n))_{ij} = [1/(\sqrt{2}\kappa)][1/(n+s+1)]\{[(2(n+s)+\tfrac{3}{2}-\lambda)/(2i-1) - \Delta_i(\delta/4)g^{(-)}_{1/2}]\,\delta_{ij} + [(\delta/2)g^{(-)}_{i+1/2}/(2i-1)]\,\delta_{ij-1} + [(\delta/2)g^{(+)}_{i-1/2}/(2i-1)]\,\delta_{ij+1}\}$$

$$(\hat{R}_{21}(n))_{ij} = -(1/\sqrt{2}\kappa)[1/(n+s+1)][1/(2n+2s+3)]$$
$$\times\{[(2(n+s+1)+\tfrac{1}{2}-\lambda)/(2i-1)+\Delta_i(\delta/4)g^{(-)}_{1/2}]\,\delta_{ij}$$
$$+\tfrac{1}{2}\,\delta g^{(-)}_{i+1/2}[1/(2i-1)]\,\delta_{ij-1}+\tfrac{1}{2}\,\delta g^{(+)}_{i-1/2}[1/(2i-1)]\,\delta_{ij+1}\}$$

$$(\hat{R}_{22}(n))_{ij} = [1/(2n+2s+3)]\,\delta_{ij}+2(n+s+1)\sum_{l=1}^{N/2}(\hat{R}_{21}(n))_{il}(\hat{R}_{12}(n))_{lj}$$

where

$$\Delta_i = \begin{cases} 1 & i=1 \\ 0 & \text{otherwise} \end{cases} \qquad i,j = 1,\ldots,N/2.$$

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